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Graded contractions of representations of orthogonal and symplectic Lie algebras with respect to their maximal parabolic subalgebras

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Abstract. Parabolic gradings of the classical simple Lie algebras $o(N,\mathbb{C})$, $(N \geqslant 5)$ and $sp(2n,\mathbb{C})$, $(n \geqslant 2)$ with complex parameters are described for all maximal parabolic subalgebras. All contractions which leave a maximal parabolic subalgebra intact and which preserve a parabolic grading (parabolic contractions of Lie algebras) are found. Contractions of the irreducible representations for each parabolic contraction of the Lie algebra are the main results of the article.

1. Introduction

The study [1] of parabolic contractions of $sl(N,\mathbb{C})$ and of its representations, including tensor products of the latter, is extended here to the remaining classical simple Lie algebras $o(N,\mathbb{C})$ and $sp(2n,\mathbb{C})$ and their representations.

The method of our investigation is the same as in [1]; we find the grading-preserving contractions of the simple Lie algebras where the gradings are parabolic, i.e. the coarsest gradings which display a maximal parabolic subalgebra as a sum of several grading subspaces of the simple Lie algebra. It turns out that for most of the maximal parabolic subalgebras of $o(N,\mathbb{C})$, $n \geqslant 5$, and $sp(2n,\mathbb{C})$, $n \geqslant 2$, the grading group involved is the cyclic group $\mathbb{Z}_5$. In a few extreme cases only, the grading group is $\mathbb{Z}_3$, as it was for $sl(N,\mathbb{C})$. Consequently, the outcomes of the contraction procedure are more numerous here than in [1].

The motivation for a study of parabolic contractions of classical simple Lie algebras is similar for all of them. These algebras are most often used in applications where one, typically, tries to bring together as many different phenomena as possible in order to understand the common basic features underlying all of them. In doing this, symmetries related by standard homomorphisms, such as inclusion of one Lie algebra in another, offer obvious avenues for such a study. However, since the pioneering work of Wigner and Inönü 30 years ago [2] a wider class of relations among symmetry algebras can be studied. The recent modifications [3–5] of Wigner's approach make the vast variety of deformations of Lie algebras more accessible to an exhaustive description by reducing the scope of a study to that of deformations preserving a (any) fixed grading by a finite Abelian group G. A complete classification of such deformations and/or contractions then becomes possible. In addition, it allows one to consider simultaneously many Lie algebras which admit a given grading. Without the latter property neither this study nor [1] would have been practical.

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Thus for all the cases included here we have only to solve one contraction problem with $G = \mathbb{Z}_5$. For $G = \mathbb{Z}_3$ our problem has already been solved in [1].

Most importantly, the insistence on preservation of a chosen grading during contraction led to a natural definition of graded contractions of representations of Lie algebras in [4]. This is a problem apparently never considered in mathematical literature.

In recent physics literature one may also point out the papers [6, 7] where deformations of representations of unitary Cayley–Klein algebras based on contractions are developed. Various other aspects of graded contractions of Lie algebras are found in [8–14].

A physicist's motivation for a study of parabolic contractions of classical Lie algebras would, typically, be related to the fact that most maximal parabolic subalgebras of classical Lie algebras are the Lie algebras of inhomogeneous transformations, i.e. a semidirect product of a large (often maximal) reductive subalgebra with an Abelian ideal of 'translations'. The mechanism of the corresponding contractions of irreducible representations of the classical Lie algebras then becomes a rich, largely unexplored source of specific information about indecomposable representations of the Lie algebra of inhomogeneous transformations.

An overview of the graded contractions of representations is in [5] and is recalled again in [1]; therefore, here we point out only the definitions. A Lie algebra L and representation acting in V, graded by the same Abelian grading group G, decompose as linear spaces into the direct sum of the grading subspaces

$$L = \sum_{g\in G} L_g \qquad V = \sum_{g\in G} V_g. \tag{1.1}$$

Simultaneous grading of L and V by the same grading group G then implies the relations

$$[L_j, L_k] \subseteq L_{j+k} \qquad L_j V_m \subseteq V_{j+m} \tag{1.2}$$

which should be read as being valid for any choice of elements of L_j, L_k and V_n.

We say that a grading *displays a subalgebra* of L if the subalgebra consists of several subspaces L_g in the decomposition (1.1) of L. A grading of L is called *parabolic* if it displays a maximal parabolic subalgebra P_λ in a minimal number of subspaces L_g.

The contracted commutator is defined using a matrix $\varepsilon = (\varepsilon_{jk})$ of contraction parameters

$$[L_j, L_k]_\varepsilon := \varepsilon_{jk}[L_j, L_k] \subseteq \varepsilon_{jk} L_{j+k}. \tag{1.3}$$

In order that the outcome is a Lie algebra L^ε, the parameters ε_{jk} must be solutions of a system of quadratic equations ((1.9) of [1]) following from the Jacobi identities. A renormalization of the elements of the subspaces, $L_j \longrightarrow a_j L'_j$, $(0 \neq a_j \in \mathbb{C})$, leads to an equivalent grading contraction

$$[L'_j, L'_k]_\varepsilon := \varepsilon_{jk}\frac{a_{j+k}}{a_j a_k}[L'_j, L'_k] \subseteq \varepsilon_{jk}\frac{a_{j+k}}{a_j a_k} L'_{j+k}. \tag{1.3a}$$

Using this freedom, we succeed in having $\varepsilon_{jk} = 0$ or 1 in all of the cases considered here.

A contraction of L is called parabolic if a maximal parabolic subalgebra P_λ of L is not deformed by the contraction.

A contracted action of L on V is defined similarly by the matrix $\psi = (\psi_{jm})$ of contraction parameters defined for a fixed contracted Lie algebra L^ε as

$$L^\varepsilon_j V_m := \psi_{jm} L_j V_m \tag{1.4}$$

and by the uncontracted action of L on V. In order for the contracted operation to be a representation of L^ε, a system of quadratic equations ((1.11) in [1]) has to be solved by ψ for a given ε.

Finally, in order that a tensor product $V \otimes V'$ of two representations of L^ε be a representation of L^ε, the contraction of $V \otimes V'$ has to be introduced as

$$(V_j \otimes V'_m)_\tau := \tau_{jm} V_j \otimes V'_m \tag{1.5}$$

where τ_{jm} are the contraction parameters, subject to a system of linear equations involving matrix elements of ψ as coefficients ((1.13) in [1]). Similarly as in (1.3a), renormalizations of the subspaces V_m allow one to simplify the form of ψ and τ.

It is convenient for our purposes to introduce the matrix $\kappa = (\kappa_{jk})$ $(j, k \in G)$ of the grading structure:

$$\kappa_{jk} = \begin{cases} 1 & \text{if } [L_j, L_k] \neq 0 \\ 0 & \text{if } [L_j, L_k] = 0. \end{cases} \tag{1.6}$$

Note that the definition implies $\kappa = \kappa^{\mathrm{T}}$. For the parabolic gradings of the classical Lie algebras there is only one κ per grading group. Namely,

$$\kappa = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \emptyset & 1 \\ 1 & 1 & \emptyset \end{pmatrix} \qquad \text{for } \mathbb{Z}_3 \tag{1.7}$$

$$\kappa = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \emptyset & 1 & 1 \\ 1 & \emptyset & \emptyset & 1 & 1 \\ 1 & 1 & 1 & \emptyset & \emptyset \\ 1 & 1 & 1 & \emptyset & 1 \end{pmatrix} \qquad \text{for } \mathbb{Z}_5. \tag{1.8}$$

The contraction matrix ε is the κ for the contracted Lie algebra. In some cases [3] not all matrix elements of ε can be made equal to 0 or 1. The appearance of $\emptyset$ matrix elements in (1.7) and (1.8) is dictated by the parabolic gradings: the corresponding subspaces commute. The grading structure κ of LV is analogous: $\kappa_{jk} = 0$ and 1 respectively for $L_j V_k = 0$ and $L_j V_k = 1$.

In these and subsequent contraction matrices the $\emptyset$ represents a zero matrix element. It is convenient to distinguish $\emptyset$ from zeros which arise in similar matrices as a result of a contraction. The rows and columns of κ (and related matrices throughout this paper) are numbered from 0 to $n-1$ for $\mathbb{Z}_n$. The upper left 2×2 corner in (1.7) and the 3×3 corner in (1.8) are the matrices κ of the appropriate maximal parabolic subalgebras of L.

In general, one could study deformations of Lie algebras starting from the Abelian algebras and deforming the $\emptyset$'s of κ into non-zero values. We exclude similar cases by the choice of our problem: a $\emptyset$ of a κ cannot be deformed to a non-zero matrix element of an ε during a contraction, i.e. an Abelian Lie algebra is not contractable any further.

We assume that the reader is familiar with the concept of weight decomposition of representation spaces of the classical Lie algebras, in particular, that he/she can calculate the system of the weights starting from the highest one.

2. Parabolic gradings and contractions of $\mathbf{o}(N, \mathbb{C})$

Let $n = [N/2]$ be the rank of $o(N, \mathbb{C})$. In this section, we first describe the parabolic gradings of the Lie algebras of $o(N, \mathbb{C})$, $N \geqslant 3$, and then the parabolic contractions of $o(N, \mathbb{C})$. The four lowest cases, $N = 3, 4, 5$ and 6, are special because of the well known isomorphisms of the Lie algebras:

$$\begin{aligned} &o(3, \mathbb{C}) \simeq sl(2, \mathbb{C}) \qquad o(4, \mathbb{C}) \simeq sl(2, \mathbb{C}) \times sl(2, \mathbb{C}) \\ &o(5, \mathbb{C}) \simeq sp(4, \mathbb{C}) \qquad o(6, \mathbb{C}) \simeq sl(4, \mathbb{C}). \end{aligned} \tag{2.1}$$

Nevertheless we include them in our considerations whenever this does not cause excessive complication of the presentation of the results.

As in [1], the maximal parabolic subalgebras P_λ of $o(N, \mathbb{C})$, and consequently the parabolic gradings, are labelled by the integer λ, where

$$1 \leqslant \lambda \leqslant n \tag{2.2}$$

which can be undestood as numbering the simple roots of $o(N, \mathbb{C})$.

The maximal parabolic subalgebras P_λ of $o(N, \mathbb{C})$ are known. For our purpose it is convenient to characterize them by the subalgebra L_0 which is the zeroth component of the parabolic grading. Unlike the $sl(N, \mathbb{C})$ case, here L_0 is not generally a maximal regular subalgebra of $o(N, \mathbb{C})$. In table 1 we indicate the maximal regular subalgebra [15] of $o(N, \mathbb{C})$ and the L_0 component of each parabolic grading.

Table 1. P_λ is a maximal parabolic subalgebra of either $o(N, \mathbb{C})$ or $sp(2n, \mathbb{C})$; $L(\alpha_\lambda)$ is the maximal reductive subalgebra of $o(N, \mathbb{C})$ or $sp(2n, \mathbb{C})$ obtained by removal of the simple root α_k from its diagram; L_0 is the zeroth component of a parabolic grading.

Algebra	P_λ	$L(\alpha_\lambda)$	L_0	Grading
$o(2n+1, \mathbb{C})$, $n \geq 1$	$1 \leq \lambda \leq n$	$o(2\lambda) \times o(2n-2\lambda+1)$	$gl(\lambda) \times o(2n-2\lambda+1)$	$\mathbb{Z}_5$
$o(2n, \mathbb{C})$, $n \geq 2$	$\lambda = 1$	$o(2) \times o(2n-2)$	$gl(1) \times o(2n-2)$	$\mathbb{Z}_3$
	$2 \geq \lambda \leq n-2$	$o(2\lambda) \times o(2n-2\lambda)$	$gl(\lambda) \times o(2n-2\lambda)$	$\mathbb{Z}_5$
	$\lambda = n-1, n$	$gl(n)$	$gl(n)$	$\mathbb{Z}_3$
$sp(2n, \mathbb{C})$, $n \geq 2$	$1 \leq \lambda \leq n-1$	$sp(2\lambda) \times sp(2n-2\lambda)$	$gl(\lambda) \times sp(2n-2\lambda)$	$\mathbb{Z}_5$
	$\lambda = n$	$gl(n)$	$gl(n)$	$\mathbb{Z}_3$

Let us now describe the maximal parabolic subalgebras P_λ and the corresponding parabolic gradings in terms of $N \times N$ matrices in a manner similar to that in section 2 of [1] for $sl(N, \mathbb{C})$. Consider the defining representation of $o(N, \mathbb{C})$:

$$o(N, \mathbb{C}) = \left\{ X \mid X \in \mathbb{C}^{N \times N}, KX + X^{\mathrm{T}}K = 0, K^{\mathrm{T}} = K, \det K \neq 0 \right\}. \tag{2.3}$$

Fix a partition (λ, μ, λ) of N:

$$2\lambda + \mu = N. \tag{2.4}$$

First let $N = 2n+1$ (simple Lie algebra of type B_n) and set

$$K = \begin{pmatrix} 0 & 0 & K_\lambda \\ 0 & K_\mu & 0 \\ K_\lambda & 0 & 0 \end{pmatrix} \qquad K_a = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix} \tag{2.5}$$

where K_a is the $a \times a$ matrix with 1 on the side diagonal and 0 elsewhere. The Lie algebra $o(N,\mathbb{C})$ consists of the matrices X with the following block structure:

$$X = \begin{pmatrix} A & B & C \\ D & E & -\tilde{B}^{\mathrm{T}} \\ G & -\tilde{D}^{\mathrm{T}} & -\tilde{A}^{\mathrm{T}} \end{pmatrix} \tag{2.6}$$

where we use the notation $\tilde{Y} = KYK$ and

$$\begin{aligned} &A, C, G \in \mathbb{C}^{\lambda\times\lambda} \qquad E \in \mathbb{C}^{\mu\times\mu} \qquad B \in \mathbb{C}^{\mu\times\lambda} \qquad D \in \mathbb{C}^{\lambda\times\mu} \\ &C = -\tilde{C}^{\mathrm{T}} \qquad E = -\tilde{E}^{\mathrm{T}} \qquad G = -\tilde{G}^{\mathrm{T}}. \end{aligned} \tag{2.7}$$

The maximal parabolic subalgebras P_λ of $o(N,\mathbb{C})$ are then represented by all the upper- (or lower-) triangular matrices of the form

$$P_\lambda = \begin{pmatrix} A & B & C \\ 0 & E & -\tilde{B}^{\mathrm{T}} \\ 0 & 0 & -\tilde{A}^{\mathrm{T}} \end{pmatrix}. \tag{2.8}$$

For all cases of N odd (and for N even, with $(N = 2n)$ and $2 \leqslant \lambda \leqslant n-2$) one can easily verify by a direct computation that the matrices

$$\begin{aligned} &L_0 = \begin{pmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & -\tilde{A}^{\mathrm{T}} \end{pmatrix} \qquad L_1 = \begin{pmatrix} 0 & B & 0 \\ 0 & 0 & -\tilde{B}^{\mathrm{T}} \\ 0 & 0 & 0 \end{pmatrix} \qquad L_2 = \begin{pmatrix} 0 & 0 & C \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &L_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ G & 0 & 0 \end{pmatrix} \qquad L_4 = \begin{pmatrix} 0 & 0 & 0 \\ D & 0 & 0 \\ 0 & -\tilde{D}^{\mathrm{T}} & 0 \end{pmatrix} \end{aligned} \tag{2.9}$$

provide a parabolic $\mathbb{Z}_5$-grading of $o(N,\mathbb{C})$ with the matrix κ given in (1.8).

Next, we describe the three remaining cases, namely $\lambda = 1, n-1$, and n for even N. For $\lambda = 1$, *conditions* (2.7) *require that both blocks* C *and* G *be zero. The* $\mathbb{Z}_5$*-grading of* (2.9) in (2.6) is thus reduced to the $\mathbb{Z}_3$-grading:

$$L_0 = \begin{pmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & -\tilde{A}^{\mathrm{T}} \end{pmatrix} \qquad L_1 = \begin{pmatrix} 0 & B & 0 \\ 0 & 0 & -\tilde{B}^{\mathrm{T}} \\ 0 & 0 & 0 \end{pmatrix} \qquad L_2 = \begin{pmatrix} 0 & 0 & 0 \\ D & 0 & 0 \\ 0 & -\tilde{D}^{\mathrm{T}} & 0 \end{pmatrix}. \tag{2.10}$$

In the case $\lambda = n$ we have $\mu = 0$, thus we obtain another $\mathbb{Z}_3$-grading:

$$X = \begin{pmatrix} A & C \\ G & -\tilde{A}^{\mathrm{T}} \end{pmatrix} \tag{2.11}$$

with

$$L_0 = \begin{pmatrix} A & 0 \\ 0 & -\tilde{A}^{\mathrm{T}} \end{pmatrix} \qquad L_1 = \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix} \qquad L_2 = \begin{pmatrix} 0 & 0 \\ G & 0 \end{pmatrix}. \tag{2.12}$$

The difference between the present case and that of $\mathrm{sl}(n, \mathbb{C})$ is in the additional conditions imposed on the form of C and G in (2.7).

Finally, in the case of $\lambda = n - 1$ of $\mathrm{o}(2n, \mathbb{C})$ we still obtain a Z_3-grading but with a more complicated block structure of the matrices representing it. In order to find L_0, L_1, and L_2, we start with the matrix K of (2.5):

$$K = \begin{pmatrix} 0 & 0 & 0 & K_{n-1} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ K_{n-1} & 0 & 0 & 0 \end{pmatrix}. \tag{2.13}$$

The structure of the blocks E, B, and D in (2.6) is then

$$X = \begin{pmatrix} A & B_1 & B_2 & C \\ D_1 & E_1 & 0 & -\tilde{B}_1^{\mathrm{T}} \\ D_2 & 0 & -E_1 & -\tilde{B}_2^{\mathrm{T}} \\ G & -\tilde{D}_2^{\mathrm{T}} & -\tilde{D}_1^{\mathrm{T}} & -\tilde{A}^{\mathrm{T}} \end{pmatrix}. \tag{2.14}$$

Note that the relations in (2.7) apply here. The sizes of the blocks are

$$A, C, G, \in \mathbb{C}^{(n-1)\times(n-1)} \qquad E_1 \in \mathbb{C} \qquad B_1, B_2 \in \mathbb{C}^{1\times(n-1)} \qquad D_1, D_2 \in \mathbb{C}^{(n-1)\times 1}. \tag{2.15}$$

The $\mathbb{Z}_3$-grading structure is evident:

$$L_0 = \begin{pmatrix} A & 0 & B_2 & 0 \\ 0 & E_1 & 0 & -\tilde{B}_2^{\mathrm{T}} \\ D_2 & 0 & -E_1 & 0 \\ 0 & -\tilde{D}_2^{\mathrm{T}} & 0 & -\tilde{A}^{\mathrm{T}} \end{pmatrix} \qquad L_1 = \begin{pmatrix} 0 & B_1 & 0 & C \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\tilde{B}_1^{\mathrm{T}} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$L_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ D_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ G & 0 & -\tilde{D}_1^{\mathrm{T}} & 0 \end{pmatrix}. \tag{2.16}$$

The three maximal parabolic subalgebras are written as

$$P_\lambda = L_0 + L_1 \qquad \lambda = 1, n-1, n. \tag{2.17}$$

The specific realization of the parabolic gradings given above may often be practical; nevertheless, it offers only a limited insight into the general structure of the problem. An abstract definition of the gradings requires an identification of the conjugacy classes of elements of order three and five of the automorphism group of the Lie algebra (the Lie group $\mathrm{O}(N)$) which are responsible for the $\mathbb{Z}_3$- and $\mathbb{Z}_5$-gradings respectively. The grading subspaces are the eigenspaces of the action of an individual element g on the Lie algebra. Two elements from the same conjugacy class give equivalent gradings. In this way, a grading is defined independently of any particular realization of the Lie algebra. Indeed, both the Lie algebra and the Lie group (hence the element g) could be described in any representation and relative to any basis.

There is a complete description of conjugacy classes of elements of finite-order in compact simple Lie groups and their representatives in any irreducible representation given in [16]; it would not be practical to reproduce it here. In order to make the connection with that theory, we indicate only those conjugacy classes responsible for parabolic gradings using the notation of [16]. In particular, a conjugacy class of elements of finite-order in a simple Lie group of rank n is completely specified by the $n+1$ non-negative integers $[s_0\ s_1\ \ldots\ s_n]$ with only a trivial common divisor. We also show some pertinent examples.

The parabolic gradings of $o(N,\mathbb{C})$ are the eigenspace decompositions produced by any elements of the following conjugacy classes:

$$\begin{aligned}
&o(2n+1,\mathbb{C}), n\geqslant 3\begin{cases}\lambda=1 & [41\ldots 0],\mathbb{Z}_5\\ 2\leqslant\lambda\leqslant n & [3\underbrace{0\ldots0}_{\lambda-1}10\ldots0],\mathbb{Z}_5\end{cases}\\
&o(2n,\mathbb{C}), n\geqslant 4\begin{cases}\lambda=1 & [21\ldots000\ldots0],\mathbb{Z}_3\\ 2\leqslant\lambda\leqslant n-2 & [3\underbrace{0\ldots0}_{\lambda-1}10\ldots0],\mathbb{Z}_5\\ \lambda=n-1 & [20\ldots0010],\mathbb{Z}_3\\ \lambda=n & [20\ldots0001],\mathbb{Z}_3.\end{cases}
\end{aligned}\tag{2.18}$$

In all of these cases the matrix g required to reproduce the $\mathbb{Z}_3$- and $\mathbb{Z}_5$-gradings described by the explicit matrices given above turns out to be the unique diagonal element of its conjugacy class. To obtain the gradings (2.9) we use

$$g=\begin{pmatrix}I_\lambda e^{2\pi i/5} & & \\ & I_{N-2\lambda} & \\ & & I_\lambda\, e^{-2\pi i/5}\end{pmatrix}. \tag{2.19a}$$

Indeed, putting $e^{2\pi i/5}=\xi$, we have

$$\begin{aligned}
gXg^{-1}&=\begin{pmatrix}\xi I_\lambda & & \\ & I_{N-2\lambda} & \\ & & \xi^{-1}I_\lambda\end{pmatrix}\begin{pmatrix}A & B & C\\ D & E & \tilde{B}^{\mathrm{T}}\\ G & -\tilde{D}^{\mathrm{T}} & -\tilde{A}^{\mathrm{T}}\end{pmatrix}\begin{pmatrix}\xi^{-1} I_\lambda & & \\ & I_{N-2\lambda} & \\ & & \xi I_\lambda\end{pmatrix}\\
&=\begin{pmatrix}A & B\xi & C\xi^2\\ D\xi^{-1} & E & -\tilde{B}^{\mathrm{T}}\xi\\ G\xi^{-2} & -\tilde{D}^{\mathrm{T}}\xi^{-1} & -\tilde{A}^{\mathrm{T}}\end{pmatrix}.
\end{aligned}\tag{2.19b}$$

The eigenvalues $1,\xi,\xi^2,\xi^3$ and ξ^4 indicate the blocks forming the grading subspaces of (2.9).

Similarly we obtain the $\mathbb{Z}_3$-gradings using

$$g=\begin{cases}\begin{pmatrix}e^{2\pi i/3} & & \\ & I_{2n-2} & \\ & & e^{-2\pi i/3}\end{pmatrix} & \text{for } \lambda=1\\[2ex]
\begin{pmatrix}I_{n-1}e^{2\pi i/6} & & & \\ & e^{-2\pi i/6} & & \\ & & e^{2\pi i/6} & \\ & & & I_{n-1}e^{-2\pi i/6}\end{pmatrix} & \text{for } \lambda=n-1\\[2ex]
\begin{pmatrix}I_n e^{2\pi i/6} & \\ & I_n e^{-2\pi i/6}\end{pmatrix} & \text{for } \lambda=n.\end{cases}\tag{2.20}$$

Let us emphasize the similarity of the $\mathbb{Z}_3$-gradings here and the parabolic gradings (2.8) of sl$(N, \mathbb{C})$ in [1]. In both cases the matrices κ coincide and the parabolic contractions are computed exactly in the same way, the result being the two non-trivial contractions of (2.15) of [1].

The correspondence between the L_0's (as given by the matrices (2.9), (2.10), (2.12) and (2.16)) and the entries in table 1 is given by the value of λ. One may observe that the matrices A and $-\tilde{A}^{\mathrm{T}}$ in the L_0's stand for the direct sum of a pair of contragradient λ-dimensional representations of gl$(\lambda, \mathbb{C})$, and that the antisymmetric matrix E is the standard representation of o$(N - 2\lambda, \mathbb{C})$.

It remains to point out the difference between the isomorphic subalgebras L_0 in P_{n-1} and P_n of o$(2n, \mathbb{C})$, $n \geqslant 3$. Using the standard identification of the representations of gl$(n, \mathbb{C})$ by the highest weights we have, in the $2n$-dimensional representation,

$$P_{n-1} \supset L_0 = (10\ldots0)(+2) \oplus (00\ldots1)(-2)$$
$$P_n \supset L_0 = (10\ldots0)(-2) \oplus (00\ldots1)(+2).$$

Having described the parabolic gradings of o$(N, \mathbb{C})$, we can now turn to the parabolic contractions. It has been pointed out already that for the $\mathbb{Z}_3$-gradings the non-trivial contractions are given by (2.14) of [1]. In the case of the $\mathbb{Z}_5$-gradings we again have to look for solutions of the system (1.9) in [1] with the additional stipulation that the equalities of the system (1.9) in [1] involving zero matrix elements of the structure matrix (1.8) should be removed from the system.

Direct computation yields the following seven non-trivial parabolic contraction matrices:

$$\begin{pmatrix} 1&1&1&\cdot&\cdot\\ 1&1&\emptyset&\cdot&\cdot\\ 1&\emptyset&\emptyset&\cdot&\cdot\\ \cdot&\cdot&\cdot&\emptyset&\emptyset\\ \cdot&\cdot&\cdot&\emptyset&\cdot \end{pmatrix} \quad \begin{pmatrix} 1&1&1&\cdot&\cdot\\ 1&1&\emptyset&\cdot&\cdot\\ 1&\emptyset&\emptyset&\cdot&\cdot\\ \cdot&\cdot&\cdot&\emptyset&\emptyset\\ \cdot&\cdot&\cdot&\emptyset&1 \end{pmatrix} \quad \begin{pmatrix} 1&1&1&\cdot&1\\ 1&1&\emptyset&\cdot&\cdot\\ 1&\emptyset&\emptyset&\cdot&\cdot\\ \cdot&\cdot&\cdot&\emptyset&\emptyset\\ 1&\cdot&\cdot&\emptyset&\cdot \end{pmatrix}$$
$$\begin{pmatrix} 1&1&1&1&\cdot\\ 1&1&\emptyset&\cdot&\cdot\\ 1&\emptyset&\emptyset&\cdot&\cdot\\ 1&\cdot&\cdot&\emptyset&\emptyset\\ \cdot&\cdot&\cdot&\emptyset&\cdot \end{pmatrix} \quad \begin{pmatrix} 1&1&1&1&1\\ 1&1&\emptyset&\cdot&\cdot\\ 1&\emptyset&\emptyset&\cdot&\cdot\\ 1&\cdot&\cdot&\emptyset&\emptyset\\ 1&\cdot&\cdot&\emptyset&\cdot \end{pmatrix} \quad \begin{pmatrix} 1&1&1&1&1\\ 1&1&\emptyset&\cdot&\cdot\\ 1&\emptyset&\emptyset&\cdot&\cdot\\ 1&\cdot&\cdot&\emptyset&\emptyset\\ 1&\cdot&\cdot&\emptyset&1 \end{pmatrix} \tag{2.21}$$
$$\begin{pmatrix} 1&1&1&1&1\\ 1&1&\emptyset&1&\cdot\\ 1&\emptyset&\emptyset&\cdot&\cdot\\ 1&1&\cdot&\emptyset&\emptyset\\ 1&\cdot&\cdot&\emptyset&\cdot \end{pmatrix}.$$

Only in the last three cases does the contracted algebra remain indecomposable.

3. Parabolic gradings and contractions of sp$(2n, \mathbb{C})$

The information about the maximal reductive subalgebras of sp$(2n, \mathbb{C})$ is provided in table 1. Let us now consider the defining $2n$-dimensional representation of the Lie algebra. We have

$$\mathrm{sp}(2n, \mathbb{C}) = \left\{X \mid X \in \mathbb{C}^{2n\times 2n}, KX + X^{\mathrm{T}}K = 0, K^{\mathrm{T}} = -K, \det K \neq 0\right\}. \tag{3.1}$$

Since $N = 2n$, we fix the partition (λ, μ) of n as

$$\lambda + \mu = n \tag{3.2}$$

and choose K of the form

$$K = \begin{pmatrix} & & K_\lambda \\ & M & \\ -K_\lambda & & \end{pmatrix} \qquad 1 \leqslant \lambda \leqslant n \tag{3.3}$$

where K_a is as in (2.5) and

$$M = \begin{pmatrix} & K_\mu \\ -K_\mu & \end{pmatrix} \in \mathbb{C}^{2\mu\times 2\mu} \qquad M = -M^{\mathrm{T}} \qquad \det M \neq 0. \tag{3.4}$$

Note that M here satisfies the constraints imposed on the matrix K in (3.1). Thus, we have the block form of a general element X of the Lie algebra sp$(2n, \mathbb{C})$:

$$X = \begin{pmatrix} A & \tilde{F}^{\mathrm{T}}M & C \\ D & E & F \\ G & -\tilde{D}^{\mathrm{T}}M & -\tilde{A}^{\mathrm{T}}K \end{pmatrix} \tag{3.5}$$

where $\tilde{X} = XK_\lambda$ and in particular $\tilde{X}^{\mathrm{T}} = K_\lambda^{\mathrm{T}}X^{\mathrm{T}} = K_\lambda X^{\mathrm{T}}$,

$$\begin{aligned} &A, C, G, \in \mathbb{C}^{\lambda\times\lambda} \qquad E \in \mathbb{C}^{\mu\times\mu} \qquad F \in \mathbb{C}^{\lambda\times 2\mu} \qquad D \in \mathbb{C}^{2\mu\times\lambda} \\ &C = \tilde{C}^{\mathrm{T}}K_\lambda \qquad G = \tilde{G}^{\mathrm{T}}K_\lambda \qquad ME + E^{\mathrm{T}}M = 0. \end{aligned} \tag{3.6}$$

In particular E represents the Lie algebra sp$(2\mu, \mathbb{C})$. The parabolic gradings are $\mathbb{Z}_5$ for $\mu > 0$:

$$\begin{aligned} &L_0 = \begin{pmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & -\tilde{A}^{\mathrm{T}}K \end{pmatrix} \qquad L_1 = \begin{pmatrix} 0 & \tilde{F}^{\mathrm{T}}M & 0 \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix} \qquad L_2 = \begin{pmatrix} 0 & 0 & C \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &L_3 = \begin{pmatrix} 0 & 0 & 0 \\ D & 0 & 0 \\ 0 & -\tilde{D}^{\mathrm{T}}M & 0 \end{pmatrix} \qquad L_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ G & 0 & 0 \end{pmatrix} \qquad \mu > 0 \end{aligned} \tag{3.7}$$

and $\mathbb{Z}_3$ for $\mu = 0$:

$$L_0 = \begin{pmatrix} A & 0 \\ 0 & -\tilde{A}^{\mathrm{T}}K \end{pmatrix} \qquad L_1 = \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix} \qquad L_2 = \begin{pmatrix} 0 & 0 \\ G & 0 \end{pmatrix}. \tag{3.8}$$

The upper- (or lower-) triangular matrices represent faithfully the maximal parabolic subalgebras P_λ of sp$(2n, \mathbb{C})$. The maximal parabolic subalgebras are

$$\begin{aligned} &P_\lambda = L_0 + L_1 + L_2 \qquad \text{for } 1 \leqslant \lambda \leqslant n-1 \\ &P_\lambda = L_0 + L_1 \qquad \text{for } \lambda = n. \end{aligned} \tag{3.9}$$

The κ matrices of the parabolic gradings of sp$(2n, \mathbb{C})$ are given by (1.7) and (1.8) respectively. Therefore, the parabolic contractions of the symplectic Lie algebras, in the case of the $\mathbb{Z}_3$-grading, are given by the matrices ε of (2.14) in [1], and by (2.21) above for the grading $\mathbb{Z}_5$.

Finally, let us identify the conjugacy classes of the elements of order three and five in the symplectic group responsible for the parabolic gradings of the symplectic Lie algebras using, again, the notation of [16], similar to what was performed in (2.18) for the orthogonal groups. One then has

$$\begin{aligned} &\mathbb{Z}_5\colon\ [3\underbrace{0\ldots0}_{\lambda-1}10\ldots0] \qquad 1\leqslant\lambda\leqslant n-1 \\ &\mathbb{Z}_3\colon\ [20\ldots001] \qquad \lambda=n. \end{aligned} \tag{3.10}$$

Any element of these conjugacy classes provides a parabolic grading of $sp(2n,\mathbb{C})$. Two elements from the same conjugacy class provide equivalent gradings. The explicit gradings exhibited in (3.7) and (3.8) are obtained by solving the eigenspace problem (2.19b) using (3.5) and g of (2.19a) for $1\leqslant\lambda\leqslant n-1$, and g of (2.20) for $\lambda=n$.

Let us now consider an example of the $sp(6,\mathbb{C})$ representation of dimension six with the highest weight (100) and find the three maximal parabolic subalgebras P_λ, $\lambda=1,2,3$.

The parabolic gradings of the Lie algebra $sp(6,\mathbb{C})$ can be described using the six-dimensional representation of a general element X of $sp(6,\mathbb{C})$:

$$X=\begin{pmatrix} h_1 & a & d & h & j & k \\ q & h_2 & b & f & g & j \\ u & r & h_3 & c & f & h \\ x & v & s & \bar{h}_3 & \bar{b} & \bar{d} \\ z & t & v & \bar{r} & \bar{h}_2 & \bar{a} \\ w & z & x & \bar{u} & \bar{q} & \bar{h}_1 \end{pmatrix} \qquad a,b,\ldots,z\in\mathbb{C} \tag{3.11}$$

following from (3.1)–(3.4). The overbar denotes a minus sign. The element X in (3.11) can be written in the block form (3.5) defined in (3.6), the size of the blocks depending on the value of λ.

Let us fix a basis of $sp(6,\mathbb{C})$ compatible with the choices made in (3.3) and (3.4). The generators of a root decomposition of $sp(2n,\mathbb{C})$ are

$$\begin{aligned} h_{\alpha_1}&=\begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\bar{1} \end{pmatrix} & h_{\alpha_2}&=\begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\bar{1}&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix} \\ h_{\alpha_3}&=\begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\bar{1}&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix} & e_{\alpha_1}&=\begin{pmatrix} \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\bar{1}\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix} \\ e_{\alpha_2}&=\begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\bar{1}&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix} & e_{\alpha_3}&=\begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix} \end{aligned} \tag{3.12}$$

where $e_{-\alpha_i} = (e_{\alpha_i})^{\mathrm{T}}$, $i = 1, 2, 3$, and their commutators.

According to (3.10) there are two maximal parabolic $\mathbb{Z}_5$-gradings for $\lambda = 1, 2$ and one $\mathbb{Z}_3$-grading for $\lambda = 3$.

The grading subspaces of the maximal parabolic subalgebras in $\mathbb{Z}_5$-gradings are written in block form as

$$L_0 = \begin{pmatrix} A & \cdot & \cdot \\ \cdot & E & \cdot \\ \cdot & \cdot & -\tilde{A}^{\mathrm{T}}K \end{pmatrix} \qquad L_1 = \begin{pmatrix} \cdot & \tilde{F}^{\mathrm{T}}M & \cdot \\ \cdot & \cdot & F \\ \cdot & \cdot & \cdot \end{pmatrix} \qquad L_2 = \begin{pmatrix} \cdot & \cdot & C \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

where the elements of (3.11) in each block are, for $\lambda = 1$,

$$A = h_1 \qquad C = k \qquad F = \begin{pmatrix} j \\ h \\ \bar{d} \\ \bar{a} \end{pmatrix} \qquad E = \begin{pmatrix} h_2 & b & f & g \\ r & h_3 & c & f \\ v & s & \bar{h}_3 & \bar{b} \\ t & v & \bar{r} & \bar{h}_2 \end{pmatrix}$$

and for $\lambda = 2$

$$A = \begin{pmatrix} h_1 & a \\ q & h_2 \end{pmatrix} \qquad C = \begin{pmatrix} j & k \\ g & j \end{pmatrix} \qquad F = \begin{pmatrix} f & h \\ \bar{b} & \bar{d} \end{pmatrix} \qquad E = \begin{pmatrix} h_3 & c \\ s & \bar{h}_3 \end{pmatrix}.$$

For $\lambda = 3$ we have the $\mathbb{Z}_3$ grading with $P_3 = L_0 + L_1$ where

$$L_0 = \begin{pmatrix} A & \cdot \\ \cdot & -\tilde{A}^{\mathrm{T}}K \end{pmatrix} \qquad L_1 = \begin{pmatrix} \cdot & C \\ \cdot & \cdot \end{pmatrix}. \tag{3.13}$$

The blocks E and F are zero in this case; thus, one is left with, for $\lambda = 3$,

$$A = \begin{pmatrix} h_1 & a & d \\ q & h_2 & b \\ u & r & h_3 \end{pmatrix} \qquad C = \begin{pmatrix} h & j & k \\ f & g & j \\ c & f & h \end{pmatrix}.$$

4. Parabolic gradings of representations

A simultaneous grading of the Lie algebra L and its representation space V means a *simultaneous decomposition of both into eigenspaces of the action of the corresponding* grading group. We have seen that in the case of parabolic gradings the group is either $\mathbb{Z}_3$ or $\mathbb{Z}_5$. Each of these groups is generated by a single element, g say, of the maximal torus of the corresponding Lie group hence it suffices to consider the decomposition of L and V into the eigenspaces of the action of g. Note that the grading of L is a special case of that of V involving the adjoint representation.

The way in which the parabolic gradings of the Lie algebras were described above is obviously not suitable for a general representation. A suitable way for our general task is offered by the description of the action of elements of finite order on V given in [16] because the unique diagonal element in each conjugacy class is particularly easily described in any representation. One only needs to know the weight system of the representation which is a textbook computational problem in Lie theory with a well known efficient solution.

Suppose the representation $\phi(L)$ of L acts in V and suppose that we have fixed the Cartan subalgebra of L so that it is represented by diagonal matrices in ϕ. Let g be the unique element of the conjugacy class $[s_0 \ldots s_n]$ represented by the diagonal matrix $\phi(g)$. To simplify the notation we write g for $\phi(g)$. Then, the weight spaces of V (eigenspaces of the Cartan subalgebra) are also eigenspaces of g. For every $X \in L_k$, $v \in V_m$ and $w = Xv$, we have the grading action of g on L and V:

$$gXg^{-1}gv = e^{2\pi i k/M} e^{2\pi i m/M} Xv = gw = e^{2\pi i (k+m)/M} w. \tag{4.1a}$$

We write, symbolically,

$$gL_k g^{-1} = e^{2\pi i k/M} L_k \qquad gV_m = e^{2\pi i m/M} V_m. \tag{4.1b}$$

The subspaces L_k and V_k coincide in the case of the adjoint representation.

There is a technical complication arising during the grading of representations that is due to the different behaviour of the centre of the Lie group in different irreducible representations. Although it is automatically taken care of in our formalism described below, it is useful to be aware of it in order to read the results correctly. The elements g of (2.18) or (3.10) have the third and fifth roots of unity as eigenvalues when acting on the (adjoint representation of the) Lie algebra or on any other representation where the centre of the Lie group is trivially represented. When acting on a general irreducible representation of $o(N, \mathbb{C})$ or $sp(2n, \mathbb{C})$, the order of g may be multiplied by a divisor of the order of the centre, i.e. two in the case of $o(N, \mathbb{C})$ or $sp(2n, \mathbb{C})$ and also four in the case of $o(2n, \mathbb{C})$. However, the number of non-zero grading subspaces in V remains the same as in the case of L, i.e. three for $\mathbb{Z}_3$ or five for $\mathbb{Z}_5$, no matter what the order of $\phi(g)$ is.

A simultaneous grading of an irreducible representation $\phi(L)$ acting in $V(L)$ implies the grading decompositions (1.1) with the property

$$0 \neq L_k V_m \subseteq V_{k+m}. \tag{4.2}$$

The grading subspaces V_m are defined by (4.1b).

We start from the known weight decomposition of V,

$$V = \sum_{\omega} V(\omega) \tag{4.3}$$

where the dimension of $V(\omega)$ is the multiplicity of the weight ω in the weight system of the representation, and the summation extends over the weights of the representation. Our task is to 'coarsen' (4.3) into (1.1). A subspace in the decomposition (1.1) of V, say V_m, consists of all the $V(\omega)$'s of V on which $\phi(g)$ has the same eigenvalue $\exp(2\pi i m/M)$. Thus, we need to know M and m in order to know the eigenvalue of g on $V(\omega)$ in (4.1b).

The general theory [16] provides the prescription for finding the eigenvalues of all $[s_0 \ \ldots \ s_n]$. For the conjugacy classes of (2.18) and (3.10) the prescription is further simplified. The simplification is due to the fact that in (2.18) and (3.10) we always have $\sum_{k=1}^{n} s_k = 1$.

The value of M, called the adjoint order of g in [16], is independent of the representation and coincides for all the g's of $[s_0 \ \ldots \ s_n]$:

$$M = s_0 + \sum_{p=1}^{n} m_p s_p. \tag{4.4}$$

Here m_p is the coefficient of α_p in the expression

$$\theta = \begin{cases} \alpha_1 + 2\alpha_2 + \ldots + 2\alpha_n & \text{for } \mathrm{o}(2n+1, \mathbb{C}) \\ \alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n & \text{for } \mathrm{o}(2n, \mathbb{C}) \\ 2\alpha_1 + \ldots + 2\alpha_{n-1} + \alpha_n & \text{for } \mathrm{sp}(2n, \mathbb{C}) \end{cases} \tag{4.5}$$

of the highest root θ of L relative to the basis of simple roots.

The value of m is determined by λ and the weight ω. Put $m = c_\lambda(\omega)$. For the parabolic gradings it turns out [16] that $c_\lambda(\omega)$ is the coefficient of the simple root α_λ in the expression for the weight ω as a linear combination of the simple roots

$$\omega = \sum_{k=1}^{n} c_k(\omega)\alpha_k. \tag{4.6}$$

For a given simple Lie algebra L, the coefficients c_k of all weights of all irreducible finite-dimensional representations can be transformed to the common denominator D equal to the determinant of the Cartan matrix of L.

Assuming that all of the c_λ's were transformed to the common denominator D, one notices that the grading group becomes $\mathbb{Z}_{MD}$, even if the number of non-empty subspaces V_m in any $\mathbb{Z}_{MD}$-grading decomposition of an irreducible V remains at most equal to M.

The behaviour of all irreducible finite-dimensional representations of $\mathrm{o}(N, \mathbb{C})$ and $\mathrm{sp}(2n, \mathbb{C})$ with respect to the parabolic grading group $\mathbb{Z}_3$ and $\mathbb{Z}_5$ generated by the elements either in (2.18) and (3.10), falls into a few categories which are relatively simple to describe. We consider them in the rest of this section. For a number of low-dimensional representations some of the subspaces V_m may be zero.

Let us recall that there is a well known algorithm for computing weights of an irreducible representation starting from the highest weight, Ω say. The weights ω of the representation are, typically, computed relative to the basis of fundamental weights, i.e. each is given as an n-tuple of integers $\omega = (l_1, l_2, \ldots, l_n)$.

4.1. Representations of o(2n + 1, ℂ)

Consider the irreducible representations of $\mathrm{o}(2n+1, \mathbb{C})$ of finite dimension with the highest weight Ω acting in the space V^Ω. In (4.6) $c_\lambda(\omega)$ is a coordinate of a weight ω relative to the basis of simple roots

$$\omega = (l_1, \ldots, l_n) = \sum_{k=1}^{n} c_k(\omega)\alpha_k \qquad l_k \in \mathbb{Z} \tag{4.7}$$

where α_k are the simple roots of $\mathrm{o}(2n+1, \mathbb{C})$. The relation between the two bases is given by

$$c_k = \sum_{j=1}^{n} T_{kj} l_j \qquad T = (T_{kj}) = \frac{1}{2} \begin{pmatrix} 2 & 2 & 2 & \ldots & 2 & 2 \\ 2 & 4 & 4 & \ldots & 4 & 4 \\ 2 & 4 & 6 & \ldots & 6 & 6 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & 4 & 6 & \ldots & 2(n-1) & 2(n-1) \\ 1 & 2 & 3 & \ldots & (n-1) & n \end{pmatrix}. \tag{4.8}$$

A parabolic grading is fixed by the value of λ. Only one of the coefficients $c_k(\omega)$ appears in the eigenvalue $\mathrm{e}^{2\pi i m/M}$ on $V(\omega)$, namely $m = c_\lambda(\omega)$. For integer $c_\lambda(\omega)$, the eigenvalues are fifth roots of unity, for half odd $c_\lambda(\omega)$ the eigenvalues are tenth roots of unity. To simplify the presentation of the results it is convenient to transform m/M to the common denominator $2M$. This implies, in particular, doubling the subscripts in $L_j \longrightarrow L_{2j}$ and reading them modulo 10.

The irreducible representations of $o(2n+1, \mathbb{C})$ split into two mutually exclusive classes, labelled by $C = 0$ and 1 which behave differently under the parabolic gradings. The highest weight $\Omega = (a_1, a_2, \ldots, a_n)$ is said to be of congruence class C:

$$C = \begin{cases} 0 & \text{if } a_n \text{ is even} \\ 1 & \text{if } a_n \text{ is odd.} \end{cases} \tag{4.9}$$

Summarizing, we have the parabolic-grading decompositions of $o(2n+1, \mathbb{C})$ and of its irreducible representations:

$$\begin{aligned} & o(2n+1, \mathbb{C}) = L_0 + L_2 + L_4 + L_6 + L_8 \\ & V^\Omega = V_0 + V_2 + V_4 + V_6 + V_8 \qquad \Omega \text{ in congruence class } 0 \\ & V^\Omega = V_1 + V_3 + V_5 + V_7 + V_9 \qquad \Omega \text{ in congruence class } 1. \end{aligned} \tag{4.10}$$

The subscripts are read modulo 10.

Some of the subspaces V_m may be zero for the lowest few representations. Since this affects the selection of the contraction equations one has to solve, we list such cases in table 2.

Table 2. Parabolic grading of the irreducible representations of $o(2n+1, \mathbb{C})$ for which some grading subspaces are zero. The subscripts are read modulo 5 and modulo 10 for representations of congruence class 0 and 1 respectively.

$o(2n+1)$	$[410\ldots0]$	$[3010\ldots0]$	$[30010\ldots0]$	$\ldots$	$[30\ldots010]$	$[30\ldots01]$
$(100\ldots00)$	$V_0+V_1+V_4$	$V_0+V_1+V_4$	$V_0+V_1+V_4$	$\ldots$	$V_0+V_1+V_4$	$V_0+V_1+V_4$
$(010\ldots00)$	—	—	—			
$\vdots$						
$(00\ldots010)$	—	—	—			
$(00\ldots001)$	V_1+V_9	$V_0+V_2+V_8$	$V_1+V_3+V_7+V_9$			
$(00\ldots002)$	$V_0+V_1+V_4$	$V_0+V_1+V_4$				
$(00\ldots003)$	$V_1+V_3+V_7+V_9$					
$(100\ldots01)$	—					
$(010\ldots01)$	—					
$\vdots$						
$(00\ldots011)$	—					

4.2. *Representations of $sp(2n, \mathbb{C})$*

Consider the irreducible representations of $sp(2n, \mathbb{C})$. Most of the conclusions of the previous subsection can easily be adapted to apply in the present case.

We replace the simple roots of $o(2n+1,\mathbb{C})$ by those of $sp(2n,\mathbb{C})$. The adjoint order M of any $[s_0\ \dots\ s_n]$ is calculated from (4.4) and (4.5). Any weight ω is given by (4.7) where

$$c_k = \sum_{j=1}^{n} T_{kj} l_j \qquad T = \frac{1}{2}\begin{pmatrix} 2 & 2 & 2 & \dots & 2 & 1 \\ 2 & 4 & 4 & \dots & 4 & 2 \\ 2 & 4 & 6 & \dots & 6 & 3 \\ \\ 2 & 4 & 6 & \dots & 2(n-2) & n-1 \\ 2 & 4 & 6 & \dots & 2(n-1) & n \end{pmatrix}. \tag{4.11}$$

There are two congruence classes of irreducible representations. The congruence class of Ω is determined by the value

$$C = \sum_{j=1}^{n} j l_j \pmod 2. \tag{4.12}$$

For $\lambda < n$ the decompositions (4.10) apply. We have

$$1 \leqslant \lambda < n:\ sp(2n,\mathbb{C}) = L_0 + L_2 + L_4 + L_6 + L_8 \pmod{10} \tag{4.13a}$$

$$\lambda = n:\ sp(2n,\mathbb{C}) = L_0 + L_2 + L_4 \pmod 6. \tag{4.13b}$$

Next we list the special cases of lowest representations which differ from the generic cases (4.11) in that some of the subspaces are missing. By direct computation one readily verifies the following cases. For $\Omega = (10\dots0)$

$$\lambda = n:\ V^{\Omega} = V_1 + V_5, \quad V_3 = \emptyset \pmod 6 \tag{4.14}$$

for $\Omega = (10\dots0), (010\dots0), \dots, (0\dots01)$

$$1 \leqslant \lambda < n:\ V^{\Omega} = V_0 + V_2 + V_8 \qquad V_4, V_6 = \emptyset \pmod{10}. \tag{4.15}$$

4.3. Representations of $o(2n,\mathbb{C})$

The last case to consider is $o(2n,\mathbb{C})$. The weights $(l_1,\dots,l_n)$ of an irreducible representation are calculated from the highest weight Ω by the standard algorithm which gives each weight relative to the basis of fundmental weights. The coordinates of a weight relative to the basis of simple roots are obtained by

$$c_j = \sum_{k=1}^{n} T_{jk} l_k \qquad T = \frac{1}{2}\begin{pmatrix} 2 & 2 & 2 & \dots & 2 & 1 & 1 \\ 2 & 4 & 4 & \dots & 4 & 2 & 2 \\ 2 & 4 & 6 & \dots & 6 & 3 & 3 \\ \\ 2 & 4 & 6 & \dots & 2(n-2) & n-2 & n-2 \\ 1 & 2 & 3 & \dots & (n-2) & n/2 & (n-2)/2 \\ 1 & 2 & 3 & \dots & (n-2) & (n-2)/2 & n/2 \end{pmatrix}. \tag{4.16}$$

We want to describe parabolic gradings of irreducible representations of $o(2n,\mathbb{C})$. Unlike the previous cases, the situation here is somewhat more complicated. We need to consider separately even and odd values of the rank n. For a fixed n there are n parabolic gradings to consider, three of them being $\mathbb{Z}_3$ (or $\mathbb{Z}_6$ or $\mathbb{Z}_{12}$) and the rest $\mathbb{Z}_5$ (or $\mathbb{Z}_{10}$). In addition, the lowest representations have more frequently than before special (non-generic) parabolic gradings where a few of the grading subspaces may be zero.

Our task is to describe the parabolic gradings of the irreducible representations of $o(2n, \mathbb{C})$ for $\lambda = 1, 2, \cdots, n$. For every weight $\omega = (l_1, \cdots, l_n)$ the coordinates l_j relative to the basis of fundamental weights are integers. The grading for a fixed value of λ is determined by the value of c_λ of (4.16) which is the coefficient in (4.6) of the simple root α_λ in each weight ω of the representation. From (4.16) we see that for $\lambda = 1, 2, \ldots, n-2$ the values of c_λ are integer and halfinteger. For $\lambda = n-1, n$ some of the c_λ's have denominator $D = 4$.

The irreducible representations of $o(2n, \mathbb{C})$ split into four congruence classes. The classes are conveniently characterized by the two-component $C = (C_1, C_2)$. Since all weights of an irreducible representation are in the same congruence class, it suffices to determine the class of the highest weight $\Omega = (a_1, \ldots, a_n)$ (given relative to the basis of fundamental weights). One has

$$n \text{ odd: } C = (a_{n-1} + a_n, 2a_1 + 2a_3 + \cdots + 2a_{n-2} + (n-2)a_{n-1} + na_n)$$

$$n \text{ even: } C = (a_{n-1} + a_n, 2a_1 + 2a_3 + \cdots + 2a_{n-3} + (n-2)a_{n-1} + na_n).$$

Here the first component C_1 is evaluated modulo 2 while C_2 is read modulo 4.

The congruence class $(0, 0)$ has all weights in all its representations with integer coefficients $c_\lambda(\omega)$. By definition the adjoint representation is of this class. Consequently the parabolic gradings $\mathbb{Z}_3$ and $\mathbb{Z}_5$ have the obvious structure:

$$\begin{aligned} &o(2n, \mathbb{C}) = L_0 + L_1 + L_2 \qquad \lambda = 1, n-1, n \\ &V^\Omega = V_0 + V_1 + V_2 \pmod 3 \\ &o(2n, \mathbb{C}) = L_0 + L_1 + L_2 + L_3 + L_4 \qquad 2 \leqslant \lambda \leqslant n-2 \\ &V^\Omega = V_0 + V_1 + V_2 + V_3 + V_4 \qquad \pmod 5. \end{aligned} \tag{4.17}$$

Next, let the rank n be even. All $c_\lambda(\omega)$ now take integer or half odd values. Doubling the grading labels whenever needed, the parabolic gradings can be listed as

$$C = (0,2): \quad \begin{aligned} &V^\Omega = V_0 + V_2 + V_4 \bmod 6 \qquad \lambda = 1 \\ &V^\Omega = V_1 + V_3 + V_5 \bmod 6 \qquad \lambda = n-1, n \\ &V^\Omega = V_0 + V_2 + V_4 + V_6 + V_8 \bmod 10 \qquad 2 \leqslant \lambda \leqslant n-2 \end{aligned} \tag{4.18}$$

$$C = (1,0): \quad \begin{aligned} &V^\Omega = V_0 + V_2 + V_4 \bmod 6 \qquad \lambda = n \\ &V^\Omega = V_1 + V_3 + V_5 \bmod 6 \qquad \lambda = 1, n-1 \\ &V^\Omega = V_0 + V_2 + V_4 + V_6 + V_8 \bmod 10 \qquad \lambda = 2, 4, \ldots < n-1 \\ &V^\Omega = V_1 + V_3 + V_5 + V_7 + V_9 \bmod 10 \qquad \lambda = 3, 5, \ldots < n-1 \end{aligned} \tag{4.19}$$

$$C = (1,2): \quad \begin{aligned} &V^\Omega = V_0 + V_2 + V_4 \bmod 6 \qquad \lambda = n-1 \\ &V^\Omega = V_1 + V_3 + V_5 \bmod 6 \qquad \lambda = 1, n \\ &V^\Omega = V_0 + V_2 + V_4 + V_6 + V_8 \bmod 10 \qquad \lambda = 2, 4, \ldots < n-1 \\ &V^\Omega = V_1 + V_3 + V_5 + V_7 + V_9 \bmod 10 \qquad \lambda = 3, 5, \ldots < n-1. \end{aligned} \tag{4.20}$$

Now let the rank n be odd. The case of representations with $\Omega \in C = (0,2)$ is the same as for even rank. For Ω in either $C = (1,1)$ or $C = (1,3)$ we obtain

$$\begin{aligned} V^{\Omega} &= V_2 + V_4 + V_6 \bmod 12 && \lambda = 1 \\ V^{\Omega} &= V_1 + V_5 + V_9 \bmod 12 && \lambda = n-1, n \\ V^{\Omega} &= V_0 + V_2 + V_4 + V_6 + V_8 \bmod 10 && \lambda = 2, 4, \ldots < n-1 \\ V^{\Omega} &= V_1 + V_3 + V_5 + V_7 + V_9 \bmod 10 && \lambda = 3, 5, \ldots < n-1. \end{aligned} \tag{4.21}$$

When the Lie algebra acts on representations graded as in (4.18)–(4.21), its grading labels may need to be multiplied by two or four in order to read them as on V (i.e. modulo 6, 10 or 12).

The cases above assume that the given representation Ω is 'sufficiently large' (i.e. generic case). Some of the representations of low dimension are special (non-generic) in the sense that several of their grading subspaces V_m may be empty. The list of irreducible representations of $o(2n,\mathbb{C})$ which are non-generic consists of the following representations for all ranks $n \geqslant 4$:

$$\begin{aligned} &(10\ldots00)\colon \lambda = 2,3 && \mathbb{Z}_5,\ V_0 + V_2 + V_4 \\ &\qquad\qquad\quad \lambda = n-1, n && \mathbb{Z}_6,\ V_1 + V_5 \\ &(0\ldots010), (0\ldots001)\colon \lambda = 1 && \mathbb{Z}_6,\ V_1 + V_5 \\ &\qquad\qquad\qquad\qquad\quad \lambda = 2 && \mathbb{Z}_{10},\ V_0 + V_2 + V_8 \\ &\qquad\qquad\qquad\qquad\quad \lambda = 3 && \mathbb{Z}_{10},\ V_1 + V_3 + V_9. \end{aligned}$$

In addition, for the $o(4,\mathbb{C})$ representations (0010) and (0001), three out of four parabolic gradings are non-generic (see examples in section 5). The parabolic gradings of the representation (00010) of $o(5,\mathbb{C})$ are non-generic for $\lambda = 1, 2, 3$ and 5, and for (00001) non-generic for the cases $\lambda = 1, 2, 3$ and 4.

5. Parabolic contractions of representations

Suppose that a parabolic grading of a classical Lie algebra L has been fixed, together with the corresponding grading of a representation space V and that a corresponding parabolic contraction L^{ε} of L is chosen. We want to construct a representation $\psi(L^{\varepsilon})$ of the contracted Lie algebra acting in V. It is economical to proceed by the grading group rather than considering simple Lie algebras separately by their types. Practically, we need the solutions ψ of equations (1.11) of [1] for the chosen ε. We assume that one knows $L_j V_m$ from the standard representation theory of the classical Lie algebras, i.e. one knows it for any choice of $v \in V_m$ and $x \in L_j$.

To solve the system of contraction equations ((1.11) in [1]) for the representations, we need to know the structure of the grading, i.e. the values of k and m of (4.2) for which $L_k V_m = 0$, in order to eliminate the equations containing the corresponding ψ_{km} from the system. For sufficiently large representations (the generic case) we always have $L_k V_m \neq 0$. The non-generic cases for $o(2n,\mathbb{C})$ are listed below.

By considering the parabolic contractions of the Lie algebras above we have, in fact, described such contractions of the adjoint representation. The structure of $\mathbb{Z}_3$- or $\mathbb{Z}_5$-gradings of the maximal parabolic gradings of the Lie algebras considered here is given in (1.8) and the solutions of the corresponding contraction equations are shown in (2.15) of [1] for the $\mathbb{Z}_3$-gradings and in (2.21) above for the $\mathbb{Z}_5$-grading.

5.1. Parabolic contractions for $\mathbb{Z}_3$-gradings

In any of these cases there are the two parabolic contractions of L given by the matrices ε of (2.15) of [1]. To find all $\psi = (\psi_{jk})$ for a given ε, one solves the system of equations (1.11) in [1] *removing* from the system the equations which contain ε_{11} and ε_{22},i.e. the $\emptyset$ matrix elements of (1.7).

Direct computation yields the results shown in (4.1*a*) and (4.1*b*) in [1] for each of the two ε's. See also the examples in section 4 of [1].

5.2. Parabolic contractions for $\mathbb{Z}_5$-gradings

In all of these cases there are seven parabolic contractions of the Lie algebra given by the matrices ε of (2.21). For each of them we need to solve the system of quadratic equations (1.11) of [1] from which the equations involving ε_{22}, ε_{33}, ε_{12}, ε_{21}, ε_{34} and ε_{43} are removed, i.e. the zero matrix elements of (1.8).

The total number of non-trivial contraction matrices ψ (i.e. with entries 0 or 1 and not all $\psi_{jk} = 0$) for each of the seven contractions (2.21) is between 200 and 900, clearly more than one would like to list. With every ψ the list of solutions to the contraction equation (1.11) in [1] also contains matrices obtained from ψ by cyclic permutation of its columns; a few of the ψ's are symmetric with respect to such permutations. If the entire quintet of ψ's in such a list were represented by one member of the quintet the short list would still contain about 100 entries. The majority of these representations of L^ε are far from faithful. One of the solutions is always $\psi = \varepsilon$.

5.3. Examples

We end this section with examples of parabolic contractions of representations. The Lie algebra o(8) has three irreducible non-equivalent representations of dimension eight with the highest weights

$$\begin{pmatrix} 1\ 0\ {}^{0}_{0} \end{pmatrix} \qquad \begin{pmatrix} 0\ 0\ {}^{0}_{1} \end{pmatrix} \qquad \begin{pmatrix} 0\ 0\ {}^{1}_{0} \end{pmatrix}. \tag{5.1}$$

The highest weights here are displayed in the form of the o(8) Dynkin diagram. The three weight systems are closely related; any one can be obtained from any other by a permutation of simple roots. More precisely

$$\begin{aligned}
&\alpha_1 \longleftrightarrow \alpha_3, \alpha_2 \longleftrightarrow \alpha_2, \alpha_4 \longleftrightarrow \alpha_4 &&\begin{pmatrix} 1\ 0\ {}^{0}_{0} \end{pmatrix} \longleftrightarrow \begin{pmatrix} 0\ 0\ {}^{1}_{0} \end{pmatrix}\\
&\alpha_1 \longleftrightarrow \alpha_4,\ \alpha_2 \longleftrightarrow \alpha_2, \alpha_3 \longleftrightarrow \alpha_3 &&\begin{pmatrix} 1\ 0\ {}^{0}_{0} \end{pmatrix} \longleftrightarrow \begin{pmatrix} 0\ 0\ {}^{0}_{1} \end{pmatrix}\\
&\alpha_3 \longleftrightarrow \alpha_4, \alpha_2 \longleftrightarrow \alpha_2, \alpha_1 \longleftrightarrow \alpha_1 &&\begin{pmatrix} 0\ 0\ {}^{1}_{0} \end{pmatrix} \longleftrightarrow \begin{pmatrix} 0\ 0\ {}^{0}_{1} \end{pmatrix}.
\end{aligned} \tag{5.2}$$

We choose two representations defined by the highest weights $(1\ 0\ {}^{0}_{0})$ and $(0\ 0\ {}^{1}_{0})$. According to (2.18) there are four parabolic gradings of o(8). The parabolic gradings of o(8) and of its representations are the decompositions into eigenspaces of the diagonal element of the conjugacy classes [21000], [30100], [20010] and [20001] for $\lambda = 1, 2, 3$ and 4 respectively.

The grading of the algebra o(8) can be found by the method used in section 2. Instead of writing the algebra in terms of a matrix representation, here we use the root spaces; the generators are then

$$h_{\alpha_1}, h_{\alpha_2}, h_{\alpha_3}, h_{\alpha_4}, e_{\pm\alpha_1}, e_{\pm\alpha_2}, e_{\pm\alpha_3}, e_{\pm\alpha_4} \tag{5.3}$$

and their commutators are

$$e_{\pm(\alpha_1+\alpha_2)}, e_{\pm(\alpha_2+\alpha_3)}, e_{\pm(\alpha_2+\alpha_4)}, e_{\pm(\alpha_1+\alpha_2+\alpha_3)}, e_{\pm(\alpha_1+\alpha_2+\alpha_4)}, e_{\pm(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}, e_{\pm(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)}. \tag{5.4}$$

We list a basis of the grading subspaces for the three $\mathbb{Z}_3$-gradings ($\lambda = 1, 3, 4$) and one $\mathbb{Z}_5$-grading $\lambda = 2$.
$\lambda = 1$, [21000], $\mathbb{Z}_3$-grading:

$$\begin{aligned}
L_0 &= \left\{e_{\pm\alpha_2}, e_{\pm\alpha_3}, e_{\pm\alpha_4}, e_{\pm(\alpha_2+\alpha_3)}, e_{\pm(\alpha_2+\alpha_4)}, e_{\pm(\alpha_2+\alpha_3+\alpha_4)}, h_{\alpha_1}, h_{\alpha_2}, h_{\alpha_3}, h_{\alpha_4}\right\}\\
L_1 &= \left\{e_{\alpha_1}, e_{\alpha_1+\alpha_2}, e_{\alpha_1+\alpha_2+\alpha_3}, e_{\alpha_1+\alpha_2+\alpha_4}, e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}, e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}\right\}\\
L_2 &= \left\{e_{-\alpha_1}, e_{-(\alpha_1+\alpha_2)}, e_{-(\alpha_1+\alpha_2+\alpha_3)}, e_{-(\alpha_1+\alpha_2+\alpha_4)}, e_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}, e_{-(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)}\right\}
\end{aligned} \tag{5.5}$$

$\lambda = 3$, [20010], $\mathbb{Z}_3$-grading:

$$\begin{aligned}
L_0 &= \left\{e_{\pm\alpha_1}, e_{\pm\alpha_2}, e_{\pm\alpha_4}, e_{\pm(\alpha_1+\alpha_2)}, e_{\pm(\alpha_2+\alpha_4)}, e_{\pm(\alpha_1+\alpha_2+\alpha_4)}, h_{\alpha_1}, h_{\alpha_2}, h_{\alpha_3}, h_{\alpha_4}\right\}\\
L_1 &= \left\{e_{\alpha_3}, e_{\alpha_2+\alpha_3}, e_{\alpha_1+\alpha_2+\alpha_3}, e_{\alpha_2+\alpha_3+\alpha_4}, e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}, e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}\right\}\\
L_2 &= \left\{e_{-\alpha_3}, e_{-(\alpha_2+\alpha_3)}, e_{-(\alpha_1+\alpha_2+\alpha_3)}, e_{-(\alpha_2+\alpha_3+\alpha_4)}, e_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}, e_{-(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)}\right\}
\end{aligned} \tag{5.6}$$

$\lambda = 4$, [20001], $\mathbb{Z}_3$- grading:

$$\begin{aligned}
L_0 &= \left\{e_{\pm\alpha_1}, e_{\pm\alpha_2}, e_{\pm\alpha_3}, e_{\pm(\alpha_1+\alpha_2)}, e_{\pm(\alpha_2+\alpha_3)}, e_{\pm(\alpha_1+\alpha_2+\alpha_3)}, h_{\alpha_1}, h_{\alpha_2}, h_{\alpha_3}, h_{\alpha_4}\right\}\\
L_1 &= \left\{e_{\alpha_4}, e_{\alpha_2+\alpha_4}, e_{\alpha_2+\alpha_3+\alpha_4}, e_{\alpha_1+\alpha_2+\alpha_4}, e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}, e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}\right\}\\
L_2 &= \left\{e_{-\alpha_4}, e_{-(\alpha_2+\alpha_4)}, e_{-(\alpha_2+\alpha_3+\alpha_4)}, e_{-(\alpha_1+\alpha_2+\alpha_4)}, e_{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}, e_{-(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)}\right\}
\end{aligned} \tag{5.7}$$

$\lambda = 2$, [30100], $\mathbb{Z}_5$-grading:

$$\begin{aligned}
L_0 &= \left\{e_{\pm\alpha_1}, e_{\pm\alpha_3}, e_{\pm\alpha_4}, h_{\alpha_1}, h_{\alpha_2}, h_{\alpha_3}, h_{\alpha_4}\right\}\\
L_1 &= \left\{e_{\alpha_2}, e_{\alpha_1+\alpha_2}, e_{\alpha_2+\alpha_3}, e_{\alpha_2+\alpha_4}, e_{\alpha_1+\alpha_2+\alpha_3}, e_{\alpha_1+\alpha_2+\alpha_4}, e_{\alpha_2+\alpha_3+\alpha_4}, e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}\right\}\\
L_2 &= \left\{e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}\right\}\\
L_3 &= \left\{e_{-(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)}\right\}\\
L_4 &= \left\{e_{-\alpha_2}\right\}.
\end{aligned} \tag{5.8}$$

In order to find the gradings of the representations $(1\,0\,{}^0_0)$ and $(0\,0\,{}^1_0)$, we consult table 3 which shows the weight systems of the two representations together with the conjugacy classes responsible for the parabolic gradings and their eigenvalues on each weight space of the representations. Bringing together the weight spaces $V(\omega)$ with the same eigenvalue of the corresponding conjugacy class we find that $(1\,0\,{}^0_0)$ is decomposed into three subspaces:

Table 3. The left column contains the weight systems of two o(8)-representations in the basis of fundamental weights and in the basis of simple roots. The eigenvalues of the grading elements (2.18) on each weight space are found in subsequent columns. We use the notation $\xi_m^a = e^{2\pi i a/m}$.

Weight System			Eigenvalues of Conjugay Classes			
			[21000]	[20010]	[20001]	[30100]
$(1\,0\,\substack{0\\0})$	$=$	$\alpha_1+\alpha_2+\frac{1}{2}\alpha_3+\frac{1}{2}\alpha_4$	ξ_3	ξ_6	ξ_6	ξ_5
$(\bar{1}\,1\,\substack{0\\0})$	$=$	$\alpha_2+\frac{1}{2}\alpha_3+\frac{1}{2}\alpha_4$	ξ_3^0	ξ_6	ξ_6	ξ_5
$(0\,\bar{1}\,\substack{1\\1})$	$=$	$\frac{1}{2}\alpha_3+\frac{1}{2}\alpha_4$	ξ_3^0	ξ_6	ξ_6	ξ_5^0
$(0\,0\,\substack{\bar{1}\\1}),(1\,0\,\substack{1\\\bar{1}})$	$=$	$-\frac{1}{2}\alpha_3+\frac{1}{2}\alpha_4,\ \frac{1}{2}\alpha_3-\frac{1}{2}\alpha_4$	$\xi_3^0,\ \xi_3^0$	$\xi_6^{-1},\ \xi_6$	$\xi_6,\ \xi_6^{-1}$	$\xi_5^0,\ \xi_5^0$
$(0\,1\,\substack{\bar{1}\\\bar{1}})$	$=$	$-\frac{1}{2}\alpha_3-\frac{1}{2}\alpha_4$	ξ_3^0	ξ_6^{-1}	ξ_6^{-1}	ξ_5^0
$(1\,\bar{1}\,\substack{0\\0})$	$=$	$-\alpha_2-\frac{1}{2}\alpha_3-\frac{1}{2}\alpha_4$	ξ_3^0	ξ_6^{-1}	ξ_6^{-1}	ξ_5^{-1}
$(\bar{1}\,0\,\substack{0\\0})$	$=$	$-\alpha_1-\alpha_2-\frac{1}{2}\alpha_3-\frac{1}{2}\alpha_4$	ξ_3^{-1}	ξ_6^{-1}	ξ_6^{-1}	ξ_5^{-1}
$(0\,0\,\substack{1\\0})$	$=$	$\frac{1}{2}\alpha_1+\alpha_2+\alpha_3+\frac{1}{2}\alpha_4$	ξ_6	ξ_3	ξ_6	
$(0\,1\,\substack{\bar{1}\\0})$	$=$	$\frac{1}{2}\alpha_1+\alpha_2+\frac{1}{2}\alpha_4$	ξ_6	ξ_3^0	ξ_6	
$(1\,\bar{1}\,\substack{0\\1})$	$=$	$\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_4$	ξ_6	ξ_3^0	ξ_6	
$(1\,0\,\substack{0\\\bar{1}}),\ (\bar{1}\,0\,\substack{0\\1})$	$=$	$\frac{1}{2}\alpha_1-\frac{1}{2}\alpha_4,\ -\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_4$	$\xi_6,\ \xi_6^{-1}$	$\xi_3^0,\ \xi_3^0$	$\xi_6^{-1},\ \xi_6$	· ‖ ·
$(\bar{1}\,1\,\substack{0\\\bar{1}})$	$=$	$-\frac{1}{2}\alpha_1-\frac{1}{2}\alpha_4$	ξ_6^{-1}	ξ_3^0	ξ_6^{-1}	
$(0\,\bar{1}\,\substack{1\\0})$	$=$	$-\frac{1}{2}\alpha_1-\alpha_2-\frac{1}{2}\alpha_4$	ξ_6^{-1}	ξ_3^0	ξ_6^{-1}	
$(0\,0\,\substack{\bar{1}\\0})$	$=$	$-\frac{1}{2}\alpha_1-\alpha_2-\alpha_3-\frac{1}{2}\alpha_4$	ξ_6^{-1}	ξ_3^{-1}	ξ_6^{-1}	

$\lambda = 1$, [21000], $\mathbb{Z}_3$-grading:

$$\begin{aligned}
V_0 &= V(\alpha_2+\tfrac{1}{2}\alpha_3+\tfrac{1}{2}\alpha_4)+V(\tfrac{1}{2}\alpha_3+\tfrac{1}{2}\alpha_4)+V(-\alpha_2-\tfrac{1}{2}\alpha_3-\tfrac{1}{2}\alpha_4)+V(-\tfrac{1}{2}\alpha_3+\tfrac{1}{2}\alpha_4)\\
&\qquad +V(\tfrac{1}{2}\alpha_3-\tfrac{1}{2}\alpha_4)+V(-\tfrac{1}{2}\alpha_3-\tfrac{1}{2}\alpha_4)+V(\tfrac{1}{2}\alpha_3-\tfrac{1}{2}\alpha_4)\\
V_1 &= V(\alpha_1+\alpha_2+\tfrac{1}{2}\alpha_3+\tfrac{1}{2}\alpha_4)\\
V_2 &= V(-\alpha_1-\alpha_2-\tfrac{1}{2}\alpha_3-\tfrac{1}{2}\alpha_4).
\end{aligned} \tag{5.9}$$

$\lambda = 2$, [30100], $\mathbb{Z}_5$- grading:

$$\begin{aligned}
V_0 &= V(\tfrac{1}{2}\alpha_3+\tfrac{1}{2}\alpha_4)+V(\tfrac{1}{2}\alpha_3-\tfrac{1}{2}\alpha_4)+V(-\tfrac{1}{2}\alpha_3+\tfrac{1}{2}\alpha_4)+V(-\tfrac{1}{2}\alpha_3-\tfrac{1}{2}\alpha_4)\\
V_1 &= V(\alpha_1+\alpha_2+\tfrac{1}{2}\alpha_3+\tfrac{1}{2}\alpha_4)+V(\alpha_2+\tfrac{1}{2}\alpha_3+\tfrac{1}{2}\alpha_4)\\
V_4 &= V(-\alpha_1-\alpha_2-\tfrac{1}{2}\alpha_3-\tfrac{1}{2}\alpha_4)+V(-\alpha_2-\tfrac{1}{2}\alpha_3-\tfrac{1}{2}\alpha_4)
\end{aligned} \tag{5.10}$$

$\lambda = 3$, [20010], $\mathbb{Z}_6$-grading:

$$\begin{aligned}
V_1 &= V(\alpha_1+\alpha_2+\tfrac{1}{2}\alpha_3+\tfrac{1}{2}\alpha_4)+V(\alpha_2+\tfrac{1}{2}\alpha_3+\tfrac{1}{2}\alpha_4)+V(\tfrac{1}{2}\alpha_3+\tfrac{1}{2}\alpha_4)\\
&\qquad +V(\tfrac{1}{2}\alpha_3-\tfrac{1}{2}\alpha_4)\\
V_3 &= \emptyset\\
V_5 &= V(-\tfrac{1}{2}\alpha_3+\tfrac{1}{2}\alpha_4)+V(-\tfrac{1}{2}\alpha_3-\tfrac{1}{2}\alpha_4)+V(-\alpha_2-\tfrac{1}{2}\alpha_3-\tfrac{1}{2}\alpha_4)\\
&\qquad +V(-\alpha_1-\alpha_2-\tfrac{1}{2}\alpha_3-\tfrac{1}{2}\alpha_4).
\end{aligned} \tag{5.11}$$

A similar $\mathbb{Z}_6$-grading is obtained for $\lambda = 4$ ([20001]) from (5.11) by the permutation $\alpha_3 \longleftrightarrow \alpha_4$.

The corresponding gradings for the representation $(0\ 0\ {}^{1}_{0})$ can be obtained either by direct computation as before or by simultaneous the permutation of certain simple roots and permutation of decompositions of V for some values of λ. More precisely we have

$$\begin{aligned}
&\lambda = 1, [21000], \mathbb{Z}_6 - \text{grading: } (5.11) \text{ with } \alpha_1 \longleftrightarrow \alpha_3\\
&\lambda = 2, [30100], \mathbb{Z}_5 - \text{grading: } (5.10) \text{ with } \alpha_1 \longleftrightarrow \alpha_3\\
&\lambda = 3, [20010], \mathbb{Z}_3 - \text{grading: } (5.9) \text{ with } \alpha_1 \longleftrightarrow \alpha_3\\
&\lambda = 4, [20001], \mathbb{Z}_6 - \text{grading: } (5.11) \text{ with } \alpha_3 \longleftrightarrow \alpha_4 \text{ followed by } \alpha_1 \longleftrightarrow \alpha_3.
\end{aligned} \tag{5.12}$$

The grading structures in the two representations of our example are nongeneric. Indeed, we find $L_j V_k = 0$ for certain values of j and k either because $V_k = 0$ or because $V_{j+k} = 0$. A concise listing of these cases is provided in the form of the corresponding matrices κ:

$$\begin{aligned}
&\mathbb{Z}_3\text{-grading: } \begin{pmatrix} 1 & 1 & 1\\ 1 & \emptyset & 1\\ 1 & 1 & \emptyset \end{pmatrix} \qquad V = V_0 + V_1 + V_2 \pmod 3\\
&\qquad\qquad\quad \begin{pmatrix} 1 & \emptyset & 1\\ \emptyset & \emptyset & 1\\ 1 & \emptyset & \emptyset \end{pmatrix} \qquad V = V_1 + V_5 \pmod 6\\
&\mathbb{Z}_5\text{-grading: } \begin{pmatrix} 1 & 1 & \emptyset & \emptyset & 1\\ 1 & \emptyset & \emptyset & \emptyset & 1\\ \emptyset & \emptyset & \emptyset & \emptyset & 1\\ \emptyset & 1 & \emptyset & \emptyset & \emptyset\\ 1 & 1 & \emptyset & \emptyset & \emptyset \end{pmatrix} \qquad V = V_0 + V_1 + V_4 \pmod 5.
\end{aligned} \tag{5.13}$$

The maximal parabolic contractions of the representations for each of the ε in both $\mathbb{Z}_3$-gradings given in (2.15) of [1] and $\mathbb{Z}_5$-gradings given in (2.21) can be found by solving equation (1.11) in [1] together with the restrictions given above. The solutions for such non-generic maximal parabolic contraction of representations of $\mathbb{Z}_3$-gradings are discussed in section 4 of [1]. The results for $\mathbb{Z}_5$-gradings are numerous so we list the solutions for just one ε:

$$\begin{aligned}
&\varepsilon = \begin{pmatrix} 1 & 1 & 1 & 1 & 1\\ 1 & 1 & \emptyset & 1 & \cdot\\ 1 & \emptyset & \emptyset & \cdot & \cdot\\ 1 & 1 & \cdot & \emptyset & \emptyset\\ 1 & \cdot & \cdot & \emptyset & \cdot \end{pmatrix} : \psi = \begin{pmatrix} x & y & \emptyset & \emptyset & z\\ \cdot & \emptyset & \emptyset & \emptyset & \cdot\\ \emptyset & \emptyset & \emptyset & \emptyset & \cdot\\ \emptyset & \cdot & \emptyset & \emptyset & \emptyset\\ \cdot & \cdot & \emptyset & \emptyset & \emptyset \end{pmatrix} \quad \begin{pmatrix} x & y & \emptyset & \emptyset & y\\ \cdot & \emptyset & \emptyset & \emptyset & \cdot\\ \emptyset & \emptyset & \emptyset & \emptyset & \cdot\\ \emptyset & y & \emptyset & \emptyset & \emptyset\\ \cdot & \cdot & \emptyset & \emptyset & \emptyset \end{pmatrix}\\
&\begin{pmatrix} x & y & \emptyset & \emptyset & x\\ \cdot & \emptyset & \emptyset & \emptyset & x\\ \emptyset & \emptyset & \emptyset & \emptyset & \cdot\\ \emptyset & \cdot & \emptyset & \emptyset & \emptyset\\ \cdot & \cdot & \emptyset & \emptyset & \emptyset \end{pmatrix} \quad \begin{pmatrix} x & y & \emptyset & \emptyset & x\\ x & \emptyset & \emptyset & \emptyset & \cdot\\ \emptyset & \emptyset & \emptyset & \emptyset & \cdot\\ \emptyset & \cdot & \emptyset & \emptyset & \emptyset\\ \cdot & \cdot & \emptyset & \emptyset & \emptyset \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & \emptyset & \emptyset & 1\\ \cdot & \emptyset & \emptyset & \emptyset & 1\\ \emptyset & \emptyset & \emptyset & \emptyset & \cdot\\ \emptyset & 1 & \emptyset & \emptyset & \emptyset\\ \cdot & 1 & \emptyset & \emptyset & \emptyset \end{pmatrix}\\
&\begin{pmatrix} 1 & 1 & \emptyset & \emptyset & 1\\ 1 & \emptyset & \emptyset & \emptyset & 1\\ \emptyset & \emptyset & \emptyset & \emptyset & 1\\ \emptyset & \cdot & \emptyset & \emptyset & \emptyset\\ \cdot & \cdot & \emptyset & \emptyset & \emptyset \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & \emptyset & \emptyset & 1\\ 1 & \emptyset & \emptyset & \emptyset & \cdot\\ \emptyset & \emptyset & \emptyset & \emptyset & \cdot\\ \emptyset & 1 & \emptyset & \emptyset & \emptyset\\ 1 & \cdot & \emptyset & \emptyset & \emptyset \end{pmatrix}
\end{aligned} \tag{5.14}$$

where x, y, z take the values zero and one independently.

Finally let us exemplify an explicit way [5] to visualize the contracted action of the grading subspaces of L on V. Let

$$V = V_0 + V_1 + V_2 + V_3 + V_4 =: \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix}. \tag{5.15}$$

The contracted action of L^ε on V can be written as

$$LV = \begin{pmatrix} \psi_{00}L_0 & \psi_{41}L_4 & \psi_{32}L_3 & \psi_{23}L_2 & \psi_{14}L_1 \\ \psi_{10}L_1 & \psi_{01}L_0 & \psi_{42}L_4 & \psi_{33}L_3 & \psi_{24}L_2 \\ \psi_{20}L_2 & \psi_{11}L_1 & \psi_{02}L_0 & \psi_{43}L_4 & \psi_{34}L_3 \\ \psi_{30}L_3 & \psi_{21}L_2 & \psi_{12}L_1 & \psi_{03}L_0 & \psi_{44}L_4 \\ \psi_{40}L_4 & \psi_{31}L_3 & \psi_{22}L_2 & \psi_{13}L_1 & \psi_{04}L_0 \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix}. \tag{5.16}$$

To finish the example we use (5.10) and the matrix elements of the last ψ of (5.14) in (5.16):

$$LV = \begin{pmatrix} L_0 & L_4 & \emptyset & \emptyset & \cdot \\ L_1 & L_0 & \emptyset & \emptyset & \cdot \\ \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\ \cdot & L_3 & \emptyset & \cdot & L_0 \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix} = \begin{pmatrix} L_0V_0 + L_4V_1 \\ L_1V_0 + L_0V_1 \\ 0 \\ 0 \\ L_3V_1 + L_0V_4 \end{pmatrix}. \tag{5.17}$$

Here both symbols $\emptyset$ and $\cdot$ stand for 0, the first one being there before contraction and the second one appearing as a result of the contraction. The subspaces V_2 and V_3 are not changed by the contraction; indeed, they were already absent from the parabolic grading (5.10) before the contraction.

If we now take the commutator of any two elements of the contracted Lie algebra and apply both sides of that equality to V according to (5.17), the equality is preserved. Hence (5.17) is a representation of the contracted Lie algebra in V.

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References

[1] Patera J and Leng X 1994 Contraction of representation of the special linear algebra with respect to their maximal parabolic subalgebras *J. Phys. A: Math. Gen.* **27** 1233–50

[2] Inönü E and Wigner E P 1953 *Proc. Natl Acad. Sci.* **39** 510

[3] De Montigny M and Patera J 1991 Discrete and continuous graded contractions of Lie algebras and superalgebras *J. Phys. A: Math. Gen.* **24** 525–47

[4] Moody R V and Patera J 1991 Discrete and continuous graded contractions of representations of Lie algebras *J. Phys. A: Math. Gen.* **24** 2227–58

[5] Patera J 1991 Graded contractions of Lie algebras, their representations and tensor products *Group Theory in Physics (AIP Conf. Proc. 266)* (New York: American Institute of Physics) pp 46–54

[6] Gromov N A 1991 The Gel'fand–Tsetlin representations of the unitary Cayley–Klein algebras *J. Math. Phys.* **32** 837–44

[7] Gromov N A 1990 *Contractions and Analytic Continuations of Classical Groups* (Syktyvkar) (in Russian)

[8] Herranz F J, de Montigny M, del Olmo M A and Santander M 1994 *J. Phys. A: Math. Gen.* **27** 2515–26

[9] de Montigny M, Patera J and Tolar J 1994 Graded contractions of kinematical groups of space time *J. Math. Phys.* **35** 405–25

[10] Bincer A M and Patera J 1993 Graded contractions of Casimir operators *J. Phys. A: Math. Gen.* **26** 5621–8

[11] King R C, Hussin A, Leng X and Patera J 1994 Graded contractions of the affine Lie algebra $A_1^{(1)}$, its representations and tensor products, and an application to the branching rule $A_1^{(1)} \subset A_1^{(1)}$ *J. Phys. A: Math. Gen.* **27** 4125–52

[12] de Montigny M 1994 Graded contractions of bilinear invariant forms of Lie algebra *J. Phys. A: Math. Gen.* **27** 4537–48

[13] de Montigny M and D Berubé 1993 The computer calculation of graded contractions of Lie algebras and their representations *Comput. Phys. Commun.* **76** 389–410

[14] de Montigny M 1993 The de Sitter-invariant differential equations and their contraction to Poincaré and Galilei *Nuovo Cimento* B **108** 1171–80

[15] Dynkin E B 1957 *Semisimple Subalgebras of Semisimple Lie Algebras (AMS Transl. Ser. 2)* vol 6 (Providence, RI: AMS) pp 111–378

[16] Moody R V and Patera J 1984 Characters of elements of finite order in Lie groups *SIAM J. Alg. Discrete Methods* **3** 359–83